Quantum Chutes and Ladders

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Introduction

Classical Chutes & Ladders and the Markov Approach

The board game Chutes and Ladders (or Snakes and Ladders) offers an excellent way to explore the concepts of chance and probability. The seemingly simple game can be modeled using Markov chains, which provide a mathematical lens with which to view its structure and flow. A fun blog article which this discussion is based on can be found here.

Chutes and Ladders is a “memoryless” game, meaning the outcome of each turn is independent of the previous ones. On each turn, rolling a six-sided die determines your movement - you have an equal probability of landing on any one of six squares ahead of you. If you’re lucky, you’ll land at the base of a ladder and move up quickly; if you’re not, you might end up at the top of a chute and slide down, but these events are completely independent of where you’ve been before.

This memoryless nature is the fundamental principle of a Markov Process, a sequence of probabilistic transitions from one state to another. When modeling Chutes and Ladders, we treat each square on the board as a “state” and transition between these states based on the roll of the die. This can be represented mathematically as a transition matrix, where each row-column intersection gives the probability of moving from one state to another. Statistics derived from the transition matrix reveal that the shortest game lasts seven moves, occurring about once in 660 games. On average, games take 39 turns, but due to skewed probability distribution, most games finish in fewer than 32 moves, with 22 being the most common number of moves.

Analyzing and understanding these probabilistic aspects of the classical game of Chutes and Ladders provides an excellent primer for considering its quantum counterpart. It helps in shaping our mindset, as we start thinking about probability distributions, state transitions, and most importantly, the concept of a “quantum superposition”, which will replace our classical die with a quantum version. As we move forward, we'll see that Quantum Chutes and Ladders is not just a game, but a fascinating playground for exploring and experiencing the principles of quantum mechanics.

Quantum Walks: The Quantum Analog of Markov Processes

The quantum extension of Markov processes are known as quantum walks. Quantum walks serve as a cornerstone for modeling the behavior of various quantum systems and have applications in algorithmic search problems. Unlike their classical counterparts, quantum walks operate in a reversible, unitary fashion. This essential difference arises due to the intrinsic nature of quantum mechanics, where transitions between states are governed by unitary operators rather than stochastic matrices.

More specifically, quantum walks differ from Markov processes in crucial ways: they employ unitary operators instead of transition matrices, introducing reversibility absent in classical systems. These unitary operators form a “quantum transition matrix” that governs state changes. Additionally, quantum walks utilize quantum superposition, allowing multiple board positions to be explored simultaneously, contrasting with the single-state nature of classical systems. This leads to parallel exploration of the state space, making quantum walks distinct and more complex than Markov processes.

While we explore quantum walks in the context of a children’s board game, the reality is that quantum walks are important for a number of potentially useful quantum algorithms. Quantum walks have been found to be the underlying structure in many quantum algorithms, including search algorithms like Grover's algorithm. These algorithms leverage the parallelism introduced by quantum superposition to search through unsorted databases more efficiently than any classical algorithm could. In this light, the shift from Chutes and Ladders to Quantum Chutes and Ladders serves as an intriguing case study. It transitions us from classical probability and Markov processes into the realm of quantum mechanics, unitary operations, and quantum walks. This quantum version opens a pathway to understand more complex quantum phenomena and algorithms.

IonQ Quantum Challenge #1

The classic game of Chutes and Ladders (or Snakes and Ladders) is a common children’s board game. The game can be effectively modeled using Markov Chains, capturing its “memoryless” essence. Today, we’re going to take this simple game and give it a quantum twist. Welcome to Quantum Chutes and Ladders!

In this challenge, you’ll transform this classical game into a quantum one by replacing the standard six-sided die with a fair quantum “coin”, and by constructing a quantum transition matrix to model the board as a discrete quantum random walk. This task will be graded across different levels of difficulty, from introductory quantum concepts to advanced quantum research.

Beginner Track

Task 1: Construct a fair “quantum coin”. What operator does this correspond to? What would a “four-sided” quantum coin look like? Give an implementation using your quantum SDK of choice? How can you verify if your two- and four-sided “quantum coin” is fair?

Task 2: Let’s start with a simplified board with no Chutes or Ladders:

Figure 1. Simplified board.

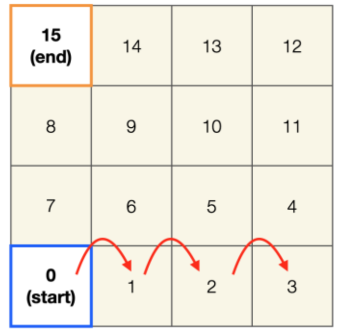

Figure 2. Simplified board and some game dynamics, moving the state one square at a time from 0 to 1 to 2…

The dynamics here are that you start at zero, and the game ends when you land on square 15. How would you represent the location on the board (4x4 grid, 16 squares) as quantum states?

Task 3: Now devise a quantum circuit to represent the game dynamics. That is, implement the quantum walk. You'll need to simulate with a quantum coin (you can use the two-sided coin) in conjunction with a "shift operator" which moves the state forward depending on the outcome of the quantum coin. Since you aren't measuring it, the board will now be in a superposition. Implement a program that evolves the gameplay for N steps of the game using your quantum SDK of choice.

Hint 1: For the dice roll and moving spaces, take some inspiration from [quantum walks and the "shift" operators]() (see [Sec 4.2.1 of this paper]()). For example, left- and right- shift operators, which increment a square by one (or decrement by one) can be given as multi-controlled Toffolis:

$$L = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \ddots & 0 \\ \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & \ddots & 0 \end{bmatrix}. \qquad (4.15)$$

Note that the L and R operators correspond to the addition and subtraction arithmetic operations, i.e., $+1$ and -1, respectively. They can be implemented by a quantum circuit consisting of multi-qubit controlled-NOT gates shown in figs. 6a and 6b. The multi-qubit controlled gates perform the carry operation in the binary format (see e.g. [Rieffel and Polak, 2011, Chapter 6]).

Figure 3. Shift operators (quantum "transition matrices") and their corresponding circuits. Schematic taken from [https://arxiv.org/abs/2203.10236]().

You can consider a unidirectional case, using either just the left or right operator. Note that because the gates mimic a cyclic graph, if you "land" on the final state and "roll again" it is possible to land back at the original state, i.e. from square 15 to square 0.

Hint 2: You will also need to consider an ancilla qubit to represent the state of the quantum coin. The wikipedia page for discrete time quantum walks may be useful.

Advanced Track

Task 4: Let's make this a little more complex. Now we will add the "chutes" and "ladders". Because the chutes and ladders must be reversible (unitary) to be quantum, each chute and ladder will permute states instead. As before, start at square zero, and the game ends when you land on square 15. If you land on square "3" you teleport to square "10". If you land on square "13", you move back to square "9". And vice versa.

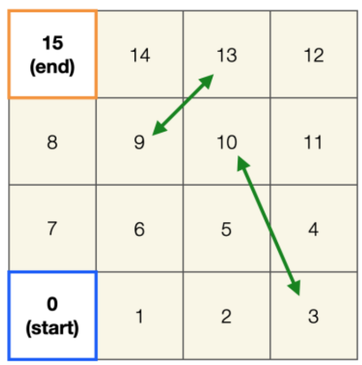

Figure 4. Board with quantum "chutes and ladders" which permute squares 3 and 10, as well as squares 9 and 13.

Hint 1: For the chutes and ladders, consider a permutation matrix, which "swaps" any two states. You may also consider implementing this with "multi-shift" operators, which move multiple spaces. To do this you will need to generalize the schematic in Figure 3.
Hint 2: Remember the product of unitary operators is unitary, so you can compose with your solution from Task 3.

Task 5: Discuss the role of measurement in your quantum Chutes and Ladders game. What happens if you measure the states between turns? What if you don't measure? What is the quantum analog of the "memoryless" nature of the classical game?

Task 6: Now, simulate your game with the Chutes and Ladders "quantum operators", similar to the walk you did in Task 3. Give a comparison of the probability of being on each square after 10 steps. How does this compare with the quantum game without the "chutes and ladders operator" in Task 3?